J. Fluid Mech. (1990), *vol.* 211, *pp.* 497–513
Printed in Great Britain

A generalized lifting-line theory for curved and swept wings

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(Received 19 September 1988 and in revised form 6 June 1989)

A generalized lifting-line theory is developed in inviscid, incompressible, steady flow for curved, swept wings of large aspect ratio. It is shown in this paper that by using the integral formulation of the problem instead of the partial differential equation formulation, it is possible to circumvent the algebraic complications encountered by the previous approaches using the method of the matched asymptotic expansions. At each approximation order the problem is reduced to inverting a classical Carleman type integral equation. The asymptotic solution in terms of circulation is found up to A^{-1} and $A^{-1}\ln(A^{-1})$. It is very convenient for illustrating the major three-dimensional effects induced on the flow by curvature and yaw angle. The concept of the finite part integrals, introduced by Hadamard (1932), is shown to be very useful for handling elegantly singularities like $1/x|x|$ or $1/|x|$ which occur in the course of our developments. Comparisons of the new, simple approach with lifting-surface theories reveal excellent agreements in terms of circulation. Furthermore, a consistent calculation of the three components of the total force acting on the wing is done in the lifting-line context without re-introducing the inner geometry of the wing.

1. Introduction

The Prandtl lifting line is the most famous and simple theory on three-dimensional wings in inviscid and incompressible flow (see Prandtl 1921). This theory is widely taught and used either to illustrate the major incompressible effects on wings or to introduce numerical methods like Vortex Lattice Methods (VLM) or panel methods. Nevertheless, the main restrictions of this model are that the wing has to be straight and has to have no yaw angle. These restrictions limit to a great extent the use of the lifting-line concept in modern applications.

Several attempts have been made to overcome these restrictions. Weissinger (1947) proposed a solution halfway between the lifting-line and the lifting-surface concepts. He employed loaded lines located at the quarter-chord line and satisfied the boundary condition at the three-quarter-chord line. This technique has been very successful for predicting spanwise loading distribution on straight, swept-back, and swept-forward wings of moderate to large aspect ratio; this method, however, cannot predict efficiently the induced drag, and is no longer valid when curvature occurs. Indeed, this technique may be seen as an early embryo of VLM.

Another way to address the problem is to use the Matched Asymptotic Expansion method (MAE). Van Dyke (1964) showed that the MAE method was relevant for treating Prandtl's lifting-line problem. He divided the flow domain into outer and inner domains. In the outer one, the lengthscale is the wing span, and the wing is modelled by a loaded line. In the inner domain, the lengthscale is the mean chord length, and the general problem degenerates into a series of two-dimensional problems. The global solution is found by matching the inner and outer expansions of the velocity potential. At about the same time as Van Dyke was proposing his new concept of the unswept lifting line, Thurber (1965) applied the asymptotic technique to curved wings of high aspect ratio. He was the first to show that the logarithm of the aspect ratio appears at the second order of the expansion rather than at the third order, as is the case for straight wings. He also pointed out the existence of an effective angle of attack, despite the presence of the logarithmic singularity. However, his presentation was rather lengthy and somewhat cumbersome. The final formulae he gave for the circulation were incorrect and so complicated that almost no physical interpretation was possible. A decade later, Cheng (1978) successfully re-applied the asymptotic technique to straight swept wings. Cheng & Murillo (1984) generalized the method to curved wings in unsteady flow. In this particular case, the MAE approach involves a great deal of algebra. The main difficulty comes from the inner solution, which is expressed in the local curvilinear coordinate system related to the lifting line. In this system of coordinates, the Laplace equation is no longer homogeneous and a particular solution of this equation is required at the second order of approximation. The calculations are even more cumbersome if the wing is curved (see Cheng & Murillo 1984).

The algebraic difficulties encountered by the MAE technique come from the desire to solve the problem from its Partial Differential Equation formulation (PDE). Indeed, when the mean line of the wing is straight and perpendicular to the flow, Kida & Miyai (1978) found a nice alternative to the MAE approach. They used the integral formulation of the problem and showed that the integral equation could be asymptotically inverted. They also proved that their solution was identical to that obtained by the MAE method. Their approach is attractive for it bypasses the non-homogeneous Laplace equation problems. In the present paper, we generalize Kida & Miyai's technique to the case of a curved wing of high aspect ratio. By using adequate mathematical tools we obtain simple, general results from which many physical interpretations can be made. To illustrate these results, comparisons are made with lifting-surface theories applied on parabolic wings and on straight, yawed, elliptical wings.

Among the mathematical tools that will be used is the concept of the finite part integral, as introduced by Hadamard (1932). This concept is useful for elegantly handling integrals with very singular kernels such as the one which occurs in the integral giving the velocity induced by a sheet of dipoles. This concept generalizes the Cauchy principal value which is normally used when the kernel behaves like $1/x$. A résumé of the relevant aspects of Hadamard's finite-part theory is given in Appendix B.

The set of forces acting on the wing are evaluated in the last section of this paper. It is shown that a consistent set of forces can be recovered within the MAE context by applying Joukowski's rule in the outer domain. The downwash induced on the lifting line in the outer model produces the same drag as that which may be recovered in the Trefftz plane. However, a side force generally appears. This inconsistency is overcome by introducing a fictitious force in the wake which can be interpreted as the force which would occur on the bound vortices if the wing were correctly modelled as a lifting surface.

2. The formulation of the problem

Let us consider a wing operating in an inviscid, incompressible flow. The velocity at infinity, along with the density, is taken as the reference. The wing is also assumed to have no thickness and a small camber ratio, so that the linearized theory can be applied. The coordinate system $(OXYZ)$ is chosen so that the OX-axis corresponds to the downstream direction, and the XOY-plane matches the mean surface of the wing. The projected surface of the wing on plane-XOY is called S and the wake is called Σ.

Let Φ be the perturbation potential of the flow, $s(X, Y)$ the camber slope and $\alpha(Y)$ the incidence of section Y. The PDE formulation of the problem is classical, and may be written as follows:

$$\left.\begin{aligned} &\Delta\Phi = 0, \\ &\frac{\partial\Phi}{\partial X} = 0 \quad \text{on } \Sigma, \\ &\frac{\partial\Phi}{\partial Z} = s(X,Y)-\alpha(Y) \quad \text{on } S, \\ &|\nabla\Phi| < \infty \quad \text{at the trailing edge,} \\ &\overset{\infty}{\Phi \to 0}. \end{aligned}\right\} \tag{1}$$

This system may be resolved by means of a Green function, and it reduces to:

$$\alpha_0(X,Y) = \frac{1}{4\pi}\mathrm{FP}\iint_{P\in S\cup\Sigma}\frac{[\![\Phi]\!]}{|MP|^3}\,\mathrm{d}\Xi\,\mathrm{d}\Psi \quad \text{for all } M(X,Y) \text{ on } S, \tag{2}$$

where $\alpha_0(X,Y) = s(X,Y)-\alpha(Y)$ and the function $[\![\Phi]\!]$ is the potential jump across surfaces S and Σ. FP before the integral signs implies the finite part in the Hadamard sense (see Hadamard 1932). The integration on the wake surface can be eliminated by integrating by parts with respect to the variable Ξ (see Ashley & Landhal 1965, §7.3 for details). Thus, (2) is simplified as follows:

$$\alpha_0(X,Y) = \frac{1}{4\pi}\mathrm{FP}\iint_S \frac{\partial[\![\Phi]\!]}{\partial\Xi}\frac{1}{(Y-\Psi)^2} \left[1+\frac{X-\Xi}{([X-\Xi]^2+[Y-\Psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\Xi\,\mathrm{d}\Psi \quad \text{for all } M(X,Y) \text{ on } S. \tag{3}$$

The Ξ-derivative of the potential jump is also known as the jump of the acceleration potential; it will be denoted $\gamma(\Xi, \Psi)$. At this point, the problem consists of inverting the integral equation, that is, the unknown function $\gamma(\Xi, \Psi)$ has to be expressed as an explicit function of $\alpha_0(X, Y)$. The modern, industrial way of treating this kind of problem is to cut S into small panels and satisfy the flow tangency condition at control points on the panels. This procedure leads to a linear system of equations which is easy to invert. However efficient this numerical technique may be, we believe that the analytical approach is useful for gaining new insight into the physical phenomena. In order to sort out the main flow characteristics we shall try to find an asymptotic solution of (3).

The lifting surface is assumed to be such that the ratio of the span lengthscale B to the chord lengthscale C is much greater than unity. This ratio is called the aspect ratio, A. The spanwise mean geometry of the wing is modelled by a smooth line L (see figure 1) whose equation is:

$$X_0 = B\,x_0(Y) \quad \text{for all} \quad M_0(X_0, Y) \quad \text{on } L, \tag{4}$$

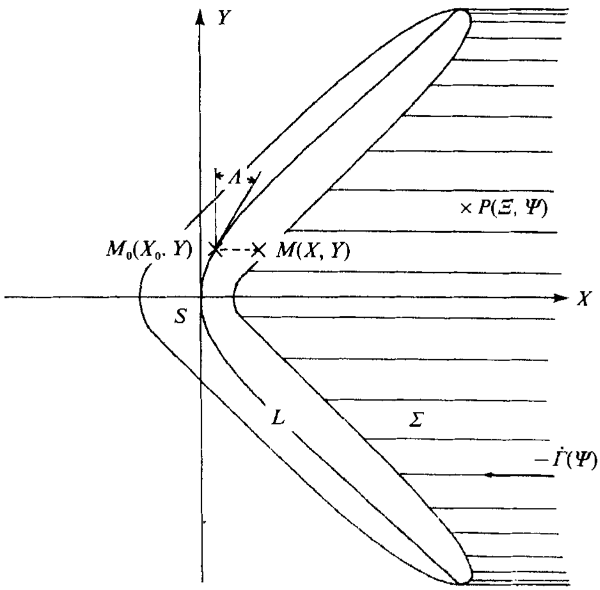

FIGURE 1. Definition of notations.

where function $x_0(Y)$ is assumed to be of order one. It is convenient to take L as a reference line in the streamwise direction, hence, the following new set of coordinates is defined:

$$\left.\begin{aligned} X &= Cx + Bx_0(y), \\ Y &= By, \\ \Xi &= C\xi + Bx_0(\psi), \\ \Psi &= B\psi. \end{aligned}\right\} \tag{5}$$

The line L is assumed to be smooth in the sense that, over the span, the function $x_0(y)$ has at least two derivatives which are of order one. Hence the non-dimensional radius of curvature is well defined over the span and is of order one. With the new coordinate system, (3) takes the form:

$$\alpha_0(x,y) = \frac{A^{-1}}{4\pi}\mathrm{FP}\iint_S \frac{\gamma(\xi,\psi)}{(y-\psi)^2}$$
$$\left[1 + \frac{(x-\xi)A^{-1} + x_0(y) - x_0(\psi)}{([(x-\xi)A^{-1} + x_0(y) - x_0(\psi)]^2 + [y-\psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\xi\,\mathrm{d}\psi \quad \text{for all } M(x,y) \text{ on } S. \tag{6}$$

3. Asymptotic expansion of downwash

At this stage, it is interesting to notice that the variable $(x-\xi)A^{-1}$ is of order A^{-1} whereas $x_0(y) - x_0(\psi)$ and $y-\psi$ are of order one. As a consequence, $(x-\xi)A^{-1}$ is much smaller, in the asymptotic sense, than the two other quantities. Therefore, it is relevant to look for an asymptotic expansion of the spanwise integral with respect to the small quantity $(x-\xi)A^{-1}$; let us change the name of this small quantity and call it ϵ. Now, the problem consists of finding an asymptotic expansion of $I(\epsilon)$:

$$I(\epsilon) = \mathrm{FP}\int_{\mathrm{Span}\,(\xi)} \frac{\mathscr{H}\gamma(\xi,\psi)}{(y-\psi)^2}\left[1 + \frac{\epsilon + x_0(y) - x_0(\psi)}{([\epsilon + x_0(y) - x_0(\psi)]^2 + [y-\psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\psi, \tag{7}$$

where $\mathscr{H}\gamma(\xi,\psi)$ is defined by:

$$\mathscr{H}\gamma(\xi,\psi) = (H[\xi - c_l(\psi)] - H[\xi - c_t(\psi)])\,\gamma(\xi,\psi). \tag{8}$$

$H[\xi]$ is the Heaviside function, and $c_l(\psi)$ and $c_t(\psi)$ are the leading-edge and the trailing-edge locations. The integration is performed along paths where ξ is constant; these ξ-lines are parallel to L. The method for finding the expansion with respect to the asymptotic sequence $\{\epsilon^j \ln(\epsilon)\}$ is given in the Appendix A. The proposed method is systematic and comes from a general study (Guermond 1987, 1988). Calculations are rather simple, since they consist of an application of a general formula (A 3). The asymptotic expansion of $I(\epsilon)$ is carried out up to $o(1)$. The result is:

$$\begin{aligned} I(\epsilon) = &-2A\int_{\mathrm{Span}\,(\xi)} \delta[\psi - y]\frac{\mathscr{H}\gamma(\xi,\psi)}{\cos(\Lambda)\,(x-\xi)}\,\mathrm{d}\psi \\ &+\mathrm{FP}\int_{\mathrm{Span}\,(\xi)} \frac{\mathscr{H}\gamma(\xi,\psi)}{(y-\psi)^2}\left[1+\frac{x_0(y)-x_0(\psi)}{([x_0(y)-x_0(\psi)]^2+[y-\psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\psi \\ &+\int_{\mathrm{Span}\,(\xi)} \delta[\psi - y]\frac{\mathscr{H}\gamma(\xi,\psi)}{r(\psi)}\left[\ln\left|\frac{x-\xi}{A}\right| + 1 - \tan^2(\Lambda) - \ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]\mathrm{d}\psi \\ &+2\int_{\mathrm{Span}\,(\xi)} \delta[\psi - y]\frac{\partial\mathscr{H}\gamma(\xi,\psi)}{\partial\psi}\left[\sin(\Lambda)\ln\left|\frac{x-\xi}{A}\right|\right. \\ &\left.+\ln\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right| - \sin(\Lambda)\ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]\mathrm{d}\psi, \end{aligned} \tag{9}$$

where $\delta[\psi]$ is the Dirac function. $r(\psi)$ is the local radius of curvature of the line L, and $\Lambda(\psi)$ is the angle between the local tangent of L at M_0 and direction $Y'OY$ (see figure 1).

The next step consists of integrating $I(\epsilon)$ with respect to ξ. The only slight difficulty in performing this integration is due to the term which contains the derivative with respect to ψ. For this term, the integration along the chord may be considered as being carried out on the interval $[c_l(\psi)+\delta, c_t(\psi)-\delta]$, where δ is a small positive quantity which converges to zero. In general terms, the following relation can be shown for all integrable function $f(\xi)$:

$$\lim_{\delta\to 0}\int_{c_l(\psi)+\delta}^{c_t(\psi)-\delta} \frac{\partial\mathscr{H}\gamma(\xi,\psi)}{\partial\psi} f(\xi)\,\mathrm{d}\xi = \frac{\partial}{\partial\psi}\int_{c_l(\psi)}^{c_t(\psi)} \gamma(\xi,\psi) f(\xi)\,\mathrm{d}\xi. \tag{10}$$

After performing the integration of $I(\epsilon)$ along the chordwise direction, the expansion of $\alpha_0(x,y)$ can be evaluated up to $o(A^{-1})$. Before going through the details of the expansion, it can be written as follows:

$$\alpha_0(x,y) = V(M_0) + V_{\mathrm{2D-vortex}} + V_{\mathrm{2D-sheet}} + o(A^{-1}) \quad \text{for all } M \text{ on } S. \tag{11}$$

The result is presented as a sum of three different terms because each of them can be given a precise interpretation. The first one has the form:

$$V(M_0) = \frac{A^{-1}}{4\pi}\mathrm{FP}\int_L \frac{\Gamma(\psi)}{(y-\psi)^2}\left[1+\frac{x_0(y)-x_0(\psi)}{([x_0(y)-x_0(\psi)]^2+[y-\psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\psi, \tag{12}$$

where $\Gamma(\psi)$ is the circulation at spanwise location ψ:

$$\Gamma(\psi) = \int_{c_l(\psi)}^{c_t(\psi)} \gamma(\xi,\psi)\,\mathrm{d}\xi. \tag{13}$$

The downwash $V(M_0)$ may easily be interpreted within the classical lifting-line context as defined by Van Dyke. Actually, the lifting-surface model would degenerate into the outer lifting-line model if $\gamma(\xi, \psi)$ were such that

$$\gamma(\xi, \psi) = \delta[\xi]\,\Gamma(\psi), \tag{14}$$

where δ is the Dirac function. Hence, if the lifting surface S were replaced by Van Dyke's outer model: the lifting line whose shape matches L and whose strength is $\Gamma(y)$, then $V(M_0)$ would be the finite part of the downwash that the lifting line would induce on itself at M_0. In other words, it would be the downwash induced at M_0 by the sheet of vortices extending downstream from L, minus the infinite contribution of the infinitesimal bound vortex along with its trailers, on which M_0 is located. This is the only downwash that would be recovered if the lifting line were straight and perpendicular to the stream. It is shown in the last section of this paper that this downwash is the one to be used to calculate the induced drag. This term comes from the outer contribution of (7), and can be recovered by setting ϵ to zero.

The last two terms of (11) come from the inner contribution of (7). The second term has the form:

$$V_{2\text{D-vortex}}(x, y) = -\frac{1}{2\pi\cos(\Lambda)}\int_{c_l(y)}^{c_t(y)} \frac{\gamma(\xi, y)}{x-\xi}\,\mathrm{d}\xi + \frac{A^{-1}}{4\pi r(y)}\left[\int_{c_l(y)}^{c_t(y)} \gamma(\xi, y)\ln\left|\frac{x-\xi}{A}\right|\mathrm{d}\xi + \Gamma(y)\left[1-\tan^2(\Lambda) - \ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]\right]. \tag{15}$$

In Van Dyke's lifting-line context, the downwash in question would reduce to:

$$V_{2\text{D-vortex}}(x, y) = -\frac{\Gamma(y)}{2\pi x\cos(\Lambda)} + \frac{A^{-1}}{4\pi r(y)}\Gamma(y)\left[\ln\left|\frac{x}{A}\right| + 1 - \tan^2(\Lambda) - \ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]. \tag{16}$$

In this form it is easy to recognize that this term is the asymptotic expansion of the velocity induced at M by the vortex ring whose strength is $\Gamma(y)$ and whose local curvature matches that of the lifting line at M_0. The presence of logarithmic terms explains why Prandtl found it impossible to extend his simplified model to curved wings. The logarithmic behaviour is also responsible for producing the non-homogeneous Laplace equation encountered by previous MAE treatments of the problem (see Cheng 1978; Cheng & Murillo 1984).

The last significant term of (11) takes the following form:

$$V_{2\text{D-sheet}}(x, y) = \frac{A^{-1}\sin(\Lambda)}{2\pi}\frac{\partial}{\partial y}\int_{c_l(y)}^{c_t(y)} \gamma(\xi, y)\ln\left|\frac{x-\xi}{A}\right|\mathrm{d}\xi + \frac{A^{-1}}{2\pi}\dot{\Gamma}(y)\left[\ln\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right| - \sin(\Lambda)\ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]. \tag{17}$$

The dot above Γ signifies the derivative with respect to y. In the lifting-line context the downwash would be written:

$$V_{2\text{D-sheet}}(x, y) = A^{-1}\frac{\dot{\Gamma}(y)}{2\pi}\left[\sin(\Lambda)\ln\left|\frac{x}{A}\right| - \sin(\Lambda)\ln\left[\frac{2}{\cos^2(\Lambda)}\right] + \ln\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right|\right]. \tag{18}$$

In this form the downwash can be interpreted as the velocity induced at M by a semi-infinite sheet of trailing vortices, whose straight and inclined upstream boundary matches the local tangent of L at M_0 and whose strength is $-\dot{\Gamma}(y)/A$. This downwash also has a logarithmic behaviour.

Indeed, the approach described above is very similar to the MAE procedure which consists of looking for the inner behaviour of the outer expression of the downwash (see Van Dyke 1964; Cheng 1978; Cheng & Murillo 1984). The advantage of the present method, however, is that the lifting surface is not degenerated into a lifting line; as a result, no fundamental information is lost. Owing to this important fact, no matching problems are encountered.

The asymptotic expansion of $I(\epsilon)$ has been restricted to $o(A^{-1})$, but the method described in Appendix A would easily give higher-order terms.

4. Asymptotic solution

In this section, it is shown that (11) can be asymptotically solved. From expansions (12), (15) and (17), it is evident that $\gamma(x,y)$ and $\Gamma(y)$ have to be approximated by:

$$\gamma(x,y) = \gamma_0(x,y) + A^{-1}\ln(A^{-1})\,\gamma_1(x,y) + A^{-1}\gamma_2(x,y) + o(A^{-1}), \tag{19}$$

$$\Gamma(y) = \Gamma_0(y) + A^{-1}\ln(A^{-1})\,\Gamma_1(y) + A^{-1}\Gamma_2(y) + o(A^{-1}). \tag{20}$$

After substituting (19) and (20) into (11), the following set of integral equations results:

$$\frac{1}{2\pi\cos(\Lambda)}\int_{c_1(y)}^{c_t(y)}\frac{\gamma_0(\xi,y)}{x-\xi}\,d\xi = -\alpha_0(x,y), \tag{21}$$

$$\frac{1}{2\pi\cos(\Lambda)}\int_{c_1(y)}^{c_t(y)}\frac{\gamma_1(\xi,y)}{x-\xi}\,d\xi = \frac{\Gamma_0(y)}{4\pi r(y)} + \frac{\dot{\Gamma}_0(y)}{2\pi}\sin(\Lambda), \tag{22}$$

$$\begin{aligned}\frac{1}{2\pi\cos(\Lambda)}\int_{c_1(y)}^{c_t(y)}\frac{\gamma_2(\xi,y)}{x-\xi}\,d\xi = {}& \frac{1}{4\pi}\left[\frac{1}{r(y)} + 2\sin(\Lambda)\frac{\partial}{\partial y}\right]\int_{c_1(y)}^{c_t(y)}\gamma_0(\xi,y)\ln|x-\xi|\,d\xi \\ &+ \frac{\Gamma_0(y)}{4\pi r(y)}\left[1-\tan^2(\Lambda) - \ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right] + W_0(M_0) \\ &+ \frac{\dot{\Gamma}_0(y)}{2\pi}\left[\ln\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right| - \sin(\Lambda)\ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right].\end{aligned} \tag{23}$$

$W_0(M_0)$ is the downwash such that:

$$W_0(M_0) = \frac{1}{4\pi}\,\mathrm{FP}\int_L \frac{\Gamma_0(\psi)}{(y-\psi)^2}\left[1 + \frac{x_0(y)-x_0(\psi)}{([x_0(y)-x_0(\psi)]^2 + [y-\psi]^2)^{\frac{1}{2}}}\right]d\psi. \tag{24}$$

The system of equations is triangular, therefore it can be progressively solved from the top to the bottom. Each equation is of the Carleman type. This kind of integral equation is very classical in two-dimensional lifting problems. If the right-hand side of the ith equation is denoted $-\alpha_i(x,y)$, then the Kutta condition at the trailing edge along with the imposed square-root behaviour of $\gamma_i(x,y)$ at the leading edge yields:

$$\gamma_i(x,y) = \frac{2\cos(\Lambda)}{\pi}\left(\frac{c_t(y)-x}{x-c_1(y)}\right)^{\frac{1}{2}}\int_{c_1(y)}^{c_t(y)}\frac{\alpha_i(\xi,y)}{x-\xi}\left(\frac{\xi-c_1(y)}{c_t(y)-\xi}\right)^{\frac{1}{2}}d\xi, \tag{25}$$

and the sectional circulation $\Gamma_i(y)$ satisfies:

$$\Gamma_i(y) = -2\cos(\Lambda)\int_{c_1(y)}^{c_t(y)}\alpha_i(\xi,y)\left(\frac{\xi-c_1(y)}{c_t(y)-\xi}\right)^{\frac{1}{2}}d\xi. \tag{26}$$

Even though the recurrent character of the present system of equations is a feature of regular perturbation problems, the present problem is actually a singular perturbation problem as discovered by Van Dyke (1964). The singular character of the perturbation $1/A$ appeared when the integral $I(\epsilon)$ was expanded with respect to ϵ (see Appendix A). The difficulties due to the singular character of ϵ have been reduced out by using the finite part concept.

In principle, there is no limit in finding higher-order solutions, since at each approximation order the right-hand side of the Carleman equation is a function of previous approximations. An example of higher-order derivation has been given in Guermond (1987) for the case of an unswept wing. In this particular case, the author showed that, for powers i and j such that $1 \leqslant j \leqslant i-2$, corrections of order $\epsilon^i \ln^j(\epsilon)$ appear. In the present case such terms will appear when $1 \leqslant j \leqslant i$.

For the first three orders, interesting simplifications occur when calculating the sectional circulation. The final result may be written in the from:

$$\Gamma_0(y) = \pi c(y) \cos(\Lambda)\,[\alpha(y) - \alpha_{01}(y)], \tag{27}$$

$$\Gamma_1(y) = \pi c(y) \cos(\Lambda) \left[\frac{\Gamma_0(y)}{4\pi r(y)} + \sin(\Lambda) \frac{\dot{\Gamma}_0(y)}{2\pi}\right], \tag{28}$$

$$\begin{aligned}\Gamma_2(y) = \pi c(y) \cos(\Lambda) \Bigg\{ & \left[\frac{\Gamma_0(y)}{4\pi r(y)} + \sin(\Lambda)\frac{\dot{\Gamma}_0(y)}{2\pi}\right] [\ln(c) - 2\ln(2) + 1 - 2K] \\ & - \frac{2}{c(y)}\left[\frac{\mathcal{M}_0(M_0)}{4\pi r(y)} + \sin(\Lambda)\frac{\dot{\mathcal{M}}_0(M_0)}{2\pi}\right] + W_0(M_0) \\ & + \frac{\Gamma_0(y)}{4\pi r(y)}\left[1 - \tan^2(\Lambda) - \ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right] \\ & + \frac{\dot{\Gamma}_0(y)}{2\pi}\left[\ln\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right| - \sin(\Lambda)\ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]\Bigg\},\end{aligned} \tag{29}$$

where $\alpha_{01}(y)$ is the classical, camber induced, zero lift incidence. K is the distance of M_0 from the leading edge expressed as a fraction of the chord; this number may not necessarily be a constant along the span. $\mathcal{M}_0(M_0)$ is the moment about M_0 of the two-dimensional distribution $\gamma_0(x, y)$, that is:

$$\mathcal{M}_0(M_0) = \int_{c_1(y)}^{c_t(y)} \xi \gamma_0(\xi, y)\,\mathrm{d}\xi. \tag{30}$$

For wings with no sectional camber slope, moment $\mathcal{M}_0(M_0)$ is given by:

$$\mathcal{M}_0(M_0) = \Gamma_0(y)\,c(y)\,[\tfrac{1}{4} - K]. \tag{31}$$

This moment is zero when M_0 is located at the quarter-chord point. It is interesting to notice that when $K = \frac{1}{4}$ the lifting-line representation of the downwash, (16) and (18), produces the same approximation of the circulation (27), (28), (29) as the lifting-surface representation.

At this stage, it is interesting to make some physical interpretations of our results. From (27), (28) and (29) it is evident that $\Gamma(y)$ may be written as follows:

$$\Gamma(y) = \pi c(y) \cos(\Lambda)\,[\alpha(y) - \alpha_{01}(y) + w(y)]. \tag{32}$$

where $w(y)$ is given by:

$$w(y) = \frac{1}{A}\left\{\left[\frac{\Gamma_0(y)}{4\pi r(y)} + \sin(\Lambda)\frac{\dot{\Gamma}_0(y)}{2\pi}\right]\left[\ln\left[\frac{c(y)}{A}\right] - 2\ln(2) + 1 - 2K\right]\right.$$
$$-\frac{2}{c(y)}\left[\frac{\mathscr{M}_0(M_0)}{4\pi r(y)} + \sin(\Lambda)\frac{\dot{\mathscr{M}}_0(M_0)}{2\pi}\right] + W_0(M_0)$$
$$+\frac{\Gamma_0(y)}{4\pi r(y)}\left[1 - \tan^2(\Lambda) - \ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]$$
$$\left.+\frac{\dot{\Gamma}_0(y)}{2\pi}\left[\ln\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right| - \sin(\Lambda)\ln\left[\frac{2}{\cos^2(\Lambda)}\right]\right]\right\}. \quad (33)$$

$W(y)$ may be interpreted as being the effective downwash in Prandtl's sense. It combines all the three-dimensional effects in one, which is equivalent to a sectional modification of the flow incidence at infinity. From this equation some classical effects induced by sweep and curvature can be qualitatively predicted. At the centre of a symmetric curved wing, the term $\Gamma_0(y)\ln[c(y)/A]/r(y)$ is dominant in the limiting case of a very high aspect-ratio wing. If the curvature opens downstream, this term produces a negative 'effective' downwash; consequently, it tends to decrease the actual loading at the centre. The effect is opposite if the curvature opens upstream. Near the tips, the term $\dot{\Gamma}(y)\sin(\Lambda)\ln[c(y)/A]$ is dominant and produces positive downwash at both tips if the wing is swept backward; thus this term tends to increase the actual loading at the tips. Therefore, both sweep and curvature shift loading from the centre towards the tips for backward sweep, and from the tips towards the centre for forward sweep. Additional arguments concerning the effects of the logarithmic downwash may be found in Cheng & Murillo (1984).

A question which may be raised at this stage concerns the choice of the function K. In other words, where should the mean line be located on the wing? In order to answer this question, let us consider a function E which represents either $\gamma(x,y)$ or $\Gamma(y)$, or whatever relevant function for which an asymptotic expansion is sought. Let E_1, E_2 be the two asymptotic expansions of E corresponding to two different mean line locations, K_1 and K_2. As long as both K_1 and K_2 are of order one and have two smooth derivatives of order one, then E_1 and E_2 satisfy equations like (19) or (20), that is:

$$E = E_1 + o(A^{-1}), \quad (34)$$

$$E = E_2 + o(A^{-1}). \quad (35)$$

Hence, E_1 differs from E_2 by terms of $o(A^{-1})$. As a result, asymptotic expansions E_1 and E_2 depend in an insignificant way, in the asymptotic sense, on particular choices of functions K_1 and K_2. The same reasoning can be carried out at higher-approximation orders if the corresponding derivatives of functions K_1 and K_2 are still of order one.

5. Comments on downwash $W_0(M_0)$

In this section, the downwash $W_0(M_0)$ is given an explicit form by the calculation of the finite part integral. If both line L and circulation $\Gamma_0(\psi)$ have simple analytical expressions, then the downwash can be analytically determined; in other cases, $W_0(M_0)$ has to be numerically evaluated. In subsequent considerations, it is assumed that the analytical work is far too complicated to be easily carried out; it is shown how numerical calculations can be undertaken.

In order to simplify subsequent mathematical expressions, an auxiliary function $f_y(\psi)$ is defined such as:

$$f_y(\psi) = \frac{\operatorname{sgn}(\psi - y)\,[x_0(\psi) - x_0(y)]}{([\psi - y]^2 + [x_0(\psi) - x_0(y)]^2)^{\frac{1}{2}}}. \tag{36}$$

As long as the line L is smooth, $f_y(\psi)$ is regular and has at least one derivative over the wing's span. It may easily be verified that:

$$f_y(y) = \sin[\Lambda(y)], \tag{37}$$

$$\dot{f}_y(y) = \frac{1}{2r(y)}. \tag{38}$$

One simple way to deal with the finite part concept is to regularize the behaviour of the integrand of (24). Let us subtract and add the first two terms of the Taylor series expansion of $\Gamma_0(\psi)$ as follows:

$$\begin{aligned} W_0(M_0) = {} & \frac{1}{4\pi}\int_L \frac{\Gamma_0(\psi) - \Gamma_0(y) - (\psi - y)\,\dot{\Gamma}_0(y)}{(y-\psi)^2}[1 - \operatorname{sgn}(\psi - y) f_y(\psi)]\,\mathrm{d}\psi \\ & + \frac{1}{4\pi}\Gamma_0(y)\,\mathrm{FP}\int_L \frac{1}{(y-\psi)^2}[1 - \operatorname{sgn}(\psi - y) f_y(\psi)]\,\mathrm{d}\psi \\ & + \frac{1}{4\pi}\dot{\Gamma}_0(y)\,\mathrm{FP}\int_L \frac{1}{(y-\psi)}[1 - \operatorname{sgn}(\psi - y) f_y(\psi)]\,\mathrm{d}\psi. \end{aligned} \tag{39}$$

The first integral is regular and can be calculated using any simple, numerical scheme such as the trapezoidal rule or Simpson's rule. The other two integrals are still defined only in the finite part sense.

Let us assume that the span lengthscale B and the origin of the coordinate system have been chosen so that the spanwise coordinate y varies between -1 and $+1$. Then, the first finite part integral of the right-hand side of (39) may be regularized as follows:

$$\begin{aligned} \mathrm{FP}\int_L \frac{1}{(y-\psi)^2}[1 - \operatorname{sgn}(\psi - y) f_y(\psi)]\,\mathrm{d}\psi = {} & -\frac{2}{1-y^2} + \frac{2y}{1-y^2}\sin[\Lambda(y)] - \frac{1}{2r(y)}\ln[1-y^2] \\ & - \int_{-1}^{+1} \frac{f_y(\psi) - f_y(y) - (\psi - y)\dot{f}_y(y)}{(\psi - y)\,|\psi - y|}\,\mathrm{d}\psi. \end{aligned} \tag{40}$$

The only integral appearing in the right-hand side is regular and can be numerically evaluated without any difficulty. The regularization procedure can be re-applied to the second finite part integral of (39); the result is:

$$\begin{aligned} & \mathrm{FP}\int_L \frac{1}{(\psi - y)}[1 - \operatorname{sgn}(\psi - y) f_y(\psi)]\,\mathrm{d}\psi \\ & \qquad = \ln\left[\frac{1-y}{1+y}\right] - \sin[\Lambda(y)]\ln[1-y^2] - \int_{-1}^{+1} \frac{f_y(\psi) - f_y(y)}{|\psi - y|}\,\mathrm{d}\psi. \end{aligned} \tag{41}$$

6. Examples and comparisons

In this section, some analytical and numerical results are presented in order to illustrate the method.

First, the circulation distribution is tested on a flat-plate, elliptical wing with a straight mid-chord line inclined to the flow at angle Λ. The aspect ratio is defined so that the non-dimensional chord-length is written as follows:

$$c(y) = \frac{2}{\cos(\Lambda)}(1-y^2)^{\frac{1}{2}}. \tag{42}$$

In this particular case, the analytical expression of the downwash $W_0(M_0)$ is quite easy to obtain. In terms of sectional circulation the final result can be put into the form:

$$\begin{aligned}\frac{\Gamma(y)}{\Gamma_0(0)} = (1-y^2)^{\frac{1}{2}}\left[1-\frac{\pi}{2A}\right]-\frac{\sin(\Lambda)}{A}(1-y^2)^{\frac{1}{2}}\arcsin(y)\\ +\frac{y\sin(\Lambda)}{A}[\ln[8A(1-y^2)^{\frac{1}{2}}]+1-2K]\\ -\frac{y}{A}[\ln[1+\sin(\Lambda)]-[1-\sin(\Lambda)]\ln[\cos(\Lambda)]]+o(A^{-1}).\end{aligned} \tag{43}$$

Function K has been assumed to be constant. This result is quite classical, and it has been shown to be in full agreement with numerical results obtained from panel methods (see Cheng 1978). The total lift coefficient has the following simple form:

$$C_L = 2\pi\alpha\cos(\Lambda)\left(1-\frac{\pi}{2A}\right). \tag{44}$$

In order to illustrate the effects of the curvature, the asymptotic solution has been tested on an elliptic flat-plate wing with a parabolic mid-chord line. The equation of the mid-line is $x_0(y) = 0.2y^2$ and the ratio of the span to the mean chord-length is 12.73. In figure 2 the distribution of the circulation with respect to the spanwise location is presented for $K = \frac{1}{2}$. The triangles are for the results obtained with a panel method devised at MIT. The agreement of the asymptotic theory (squares) with lifting-surface results is quite uniform. The asymptotic theory presents the expected improvement from the strip theory (line). Different locations of the lifting line on the

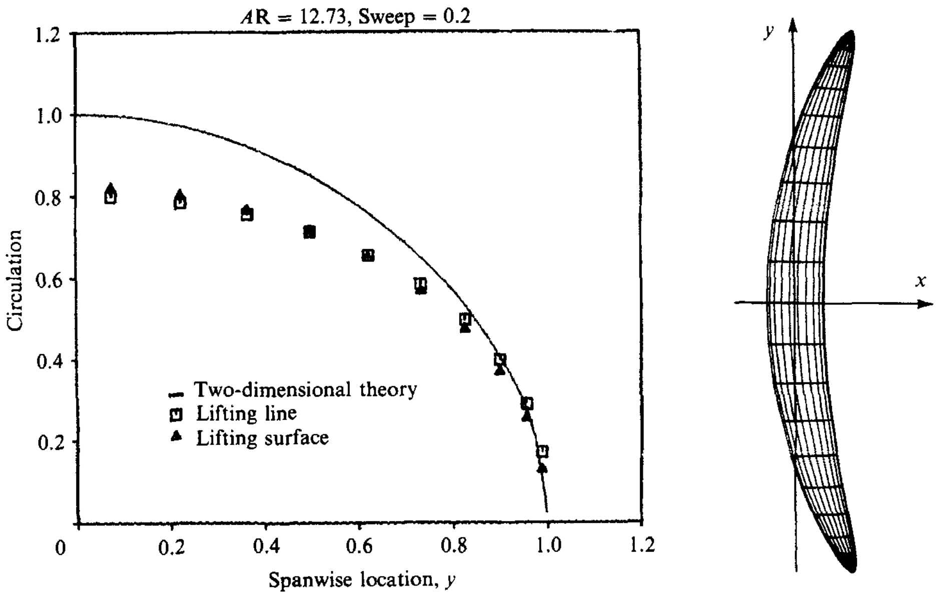

FIGURE 2. Circulation distribution on a parabolic wing.

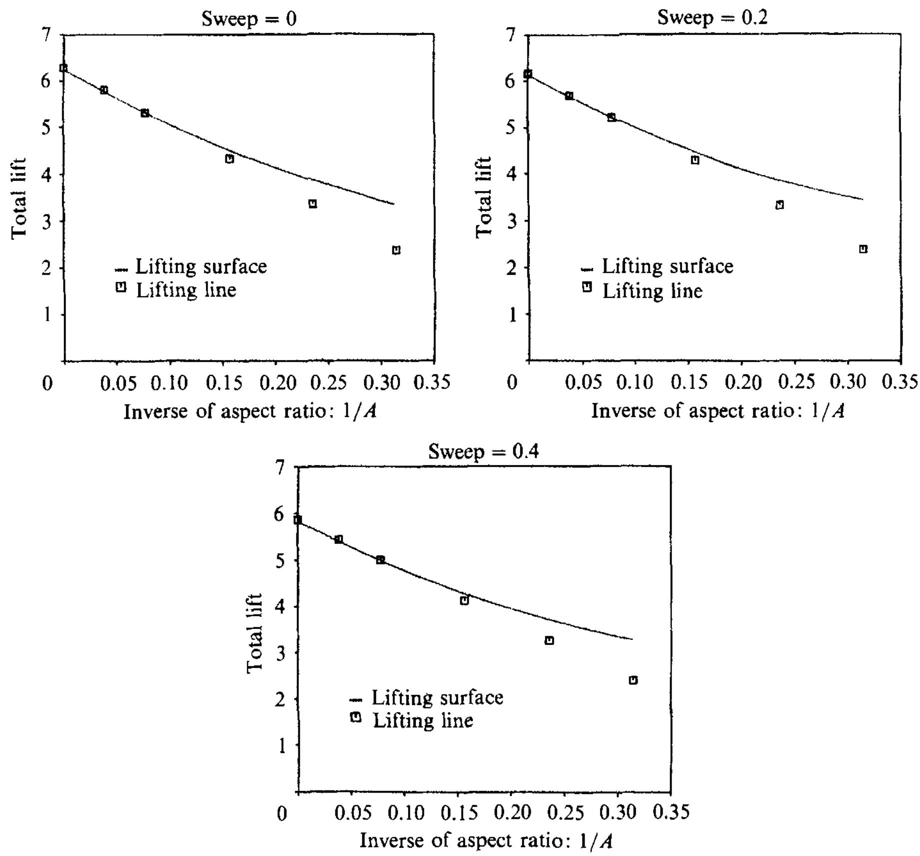

FIGURE 3. Lift coefficient as a function of sweep and A.

chord have been tried. The results (not reported here) showed that, as expected, parameter K has little effect on the results as long as the aspect ratio is asymptotically large.

In order to test both the influence of curvature and aspect ratio on total lift, calculations have been carried out on an elliptical, flat plate wing with a parabolic mid-chord line whose equation is: $x_0(y) = \text{Sweep} \times y^2$. The aspect ratio is defined as the ratio of the total span to the mean chord-length. In figure 3 lifting-line results are compared with lifting-surface results for three different values of the Sweep parameter. Coefficient K has been set to $\frac{1}{2}$. The figures clearly demonstrate that the curvature effects are uniformly predicted by the asymptotic theory. Furthermore, as already observed by Van Dyke, the first-order theory diverges for low aspect-ratio wings (see Van Dyke 1964).

7. Side-force paradox

In this last section the forces acting on the wing are considered. The evident way to calculate these forces is to integrate the pressure on the surface of the wing, then add the leading-edge suction. We assume that these operations do not pose any difficulty and shall concentrate our efforts on the lifting-line aspect of the problem. We shall show that it is possible to recover the set of forces acting on the wing using only the outer model of the lifting line.

As long as the wake geometry is frozen and planar, the circulation distribution being given, it is easy to prove, by applying the momentum theorem in the Trefftz plane, that the three components of the resultant force are independent of the wing shape, and consequently of the lifting-line geometry associated with it. Therefore the lift, drag, and side force are expressed by the well-known formulae:

$$F_L = \int_L \Gamma_0(y)\,\mathrm{d}y, \tag{45}$$

$$D = \frac{A^{-1}}{4\pi}\,\mathrm{FP}\int_L\int_L \frac{\Gamma_0(y)\,\dot{\Gamma}_0(\psi)}{y-\psi}\,\mathrm{d}y\,\mathrm{d}\psi, \tag{46}$$

$$F_S = 0. \tag{47}$$

Another classical procedure for calculating these forces is to apply Joukowski's theorem on the wing. This procedure applied to the outer lifting-line model gives both the expected lift (45) and drag (46). This result is straightforward for the lift, but needs some explanation for the drag. In order to calculate the induced drag, we need the value of the downwash induced on each infinitesimal element of the lifting line minus the (infinite) contribution of this element on itself. This required downwash is exactly $V_0(M_0)$ introduced in (12). Then, according to this result and Joukowski's rule, the drag is:

$$D = -\frac{A^{-1}}{4\pi}\,\mathrm{FP}\int_L\int_L \frac{\Gamma_0(\psi)\,\Gamma_0(y)}{(y-\psi)^2}\left[1+\frac{x_0(y)-x_0(\psi)}{([x_0(y)-x_0(\psi)]^2+[y-\psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\psi\,\mathrm{d}y. \tag{48}$$

The expected result is obtained after integrating by parts on the span with respect to ψ. Note that, to demonstrate this result, we have used the fact that the integral of an antisymmetric function over a square domain is zero.

The calculation of the side force is a little more complicated because we face a kind of paradox. Let us illustrate this with a straight, yawed lifting line. Let Λ be the angle between the mean line and the flow direction. If we apply Joukowski's theorem on it, then a side force proportional to the drag and to $\tan(\Lambda)$ occurs. This paradoxical result means that the outer model of the lifting-line approximation is partially inconsistent. One way to recover consistency is to admit that a fictitious side force is produced in the wake. In order to demonstrate this surprising result, let us consider a semi-infinite strip of trailing vortices of width $\mathrm{d}y$. Then, let us calculate the induced velocity along this strip and apply Joukowski's rule. The force per unit length acting on the semi-infinite strip is:

$$\frac{\mathrm{d}F_{S\,\mathrm{wake}}}{\mathrm{d}y} = -A^{-1}\frac{\dot{\Gamma}_0(y)}{4\pi}\,\mathrm{FP}\int_{x_0(y)}^{+\infty}\mathrm{d}t\int_L \frac{\Gamma_0(\psi)}{(y-\psi)^2}\left[1+\frac{t-x_0(\psi)}{([t-x_0(\psi)]^2+[y-\psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\psi\,\boldsymbol{j}. \tag{49}$$

The integral in the finite part sense on t is easy to evaluate; the result is:

$$\frac{\mathrm{d}F_{S\,\mathrm{wake}}}{\mathrm{d}y} = A^{-1}\frac{\dot{\Gamma}_0(y)}{4\pi} \times \mathrm{FP}\int_L \frac{\Gamma_0(\psi)}{(y-\psi)^2}[x_0(y)+x_0(\psi)+([x_0(y)-x_0(\psi)]^2+[y-\psi]^2)^{\frac{1}{2}}]\,\mathrm{d}\psi\,\boldsymbol{j}. \tag{50}$$

This force is parallel to the $Y'OY$ axis. It can be interpreted as the force which would act on the longitudinal bound vortices of each y-section of the wing, if this latter were correctly modelled as a lifting surface. The total force is recovered by performing an integration over the span and an integration by parts with respect to y, with the following result:

$$F_{\mathrm{S\,wake}} = -\frac{A^{-1}}{4\pi} \times \mathrm{FP}\int_L\int_L \frac{\Gamma_0(y)\,\Gamma_0(\psi)}{(y-\psi)^2}\frac{\mathrm{d}x_0(y)}{\mathrm{d}y}\left[1+\frac{x_0(y)-x_0(\psi)}{([x_0(y)-x_0(\psi)]^2+[y-\psi]^2)^{\frac{1}{2}}}\right]\mathrm{d}\psi\,\mathrm{d}y\boldsymbol{j}. \qquad (51)$$

In this formula, it is now easy to see that $F_{\mathrm{S\,wake}}$ is exactly opposite to the y-component of the force acting on the lifting line, which is the integral of the cross product of $V_0(M_0)$ with the local vorticity $\Gamma(y)\,(\mathrm{d}x_0, \mathrm{d}y)^{\mathrm{T}}$. Thus, the side-force 'paradox' is resolved if the fictitious side force, produced in the frozen wake, is included in the balance of the forces acting on the wing.

8. Conclusions

In this paper, we have generalized Kida & Miyai's lifting-line theory to curved and swept wings. By starting from the complete lifting-surface integral equation and by using the finite part integral concept, we have come up with a Carleman type integral equation at each approximation order. The asymptotic solution has been given both in terms of pressure jump and sectional circulation up to $o(A^{-1})$. It has been shown that the asymptotic solution depends in an insignificant way on the particular choice of the mean line location.

The new model is very simple and involves relatively little algebra compared with other MAE approaches. Higher-approximation orders are available by extending the expansion of $I(\epsilon)$ (see equation (7)). Furthermore, the simplicity of this model allows simple extensions to more complicated geometries like skewed marine propellers or windmills. The extension of this model to unsteady flow is under way and almost completed.

The basic effects on the flow induced by yaw-angle and curvature can easily be recovered from the present solution. Comparisons between this theory and panel methods reveal excellent agreements when the aspect ratio is high and the local curvature moderate. In terms of forces, the outer model of the lifting line, in the sense of MAE, gives consistent lift and induced drag, but produces a side force which cancels out if we admit the existence of a fictitious side force in the wake, which would really act on the wing if either the inner model were re-introduced or the initial lifting-surface model were retained.

The author wishes to express his sincere thanks to Professors J. E. Kerwin and P. Sclavounos for helpful discussions and for their interest in this problem. B. King and A. Sellier are also thanked for their comments and the help they provided during the preparation of the manuscript. This study has been supported by the French Navy, while the author was visiting the Department of Ocean Engineering at MIT.

Appendix A

In this Appendix we shall give a general solution for finding an asymptotic expansion with respect to ϵ of integrals such as:

$$I(\epsilon) = \mathrm{FP}\int_D f(t)\,K(t, g(t), \epsilon)\,\mathrm{d}t. \qquad (\mathrm{A}\,1)$$

D is an open real interval which contains zero: the integral is defined by its finite part. Functions $f(t)$ and $g(t)$ are integrable on D and have derivatives of all orders throughout a neighbourhood of zero; furthermore, $g(x)$ is assumed to be zero at $t=0$. Kernel K is assumed to be homogeneous of order β, which is to say:

$$K(\alpha t, \alpha g(t)\, \alpha\epsilon) = [\alpha S(\alpha)]^{\beta} K(t, g(t), \epsilon) \quad \text{for all } \alpha \text{ in } \mathbb{R}, \tag{A 2}$$

where function $S(\alpha)$ is either the sign function or the unit function; β is a real and is called the homogeneity order of the kernel. $K(t, g(t), 1)$ is assumed to be a smooth function throughout $\mathbb{R}^*$. At $t=0, K(t,g(t),1)$ may be infinite, but must have the same asymptotic behaviour, with respect to t, on both sides of zero. $K(t, g(t), \epsilon)$ has partial derivatives of all orders, with respect to ϵ, throughout a neighbourhood of zero. These conditions are sufficient for applying general results which have been demonstrated in Guermond (1987, 1988). The important result is that $I(\epsilon)$ may be approximated by:

$$\begin{aligned} I(\epsilon) = {} & \sum_{j=0}^{J} \mathrm{FP} \int_D f(t)\, \partial_3^j K(t, g(t), 0)\, \mathrm{d}t \frac{\epsilon^j}{j!} \\ & + \sum_{i=0}^{J-[\beta]-1} \frac{f^{(i)}(0)}{i!} \sum_{l=0}^{J-[\beta]-1} \sum_{m=0}^{J-[\beta]+l-1} \frac{H_{ilm}}{m!} \\ & \times \mathrm{FP} \int_{-\infty}^{+\infty} t^m \frac{\partial_2^l K(t, t\dot{g}(0), 1)}{l!} \mathrm{d}t\, \epsilon^{m-l+1} [\epsilon S(\epsilon)]^{\beta} \operatorname{sgn}(\epsilon) \\ & - R(\beta)\, [1-[S(-1)]^{[\beta]}] \sum_{i=0}^{J-[\beta]-1} \frac{f^{(i)}(0)}{i!} \sum_{l=0}^{J-[\beta]-1} \sum_{\substack{m \geqslant l-1-[\beta] \\ m \geqslant 0}}^{J-[\beta]+l-1} \\ & \times \frac{H_{ilm}}{m!} \frac{\partial_2^l \partial_3^{1+[\beta]-l+m} K(1, \dot{g}(0), 0)}{l!\,(1+[\beta]-l+m)!} \epsilon^{1+[\beta]-l+m} \ln|\epsilon| + o(\epsilon^J), \end{aligned} \tag{A 3}$$

where $\partial_l^u K$ is for the uth partial derivative of K with respect to the lth variable. $R(\beta)$ is either equal to one if β is integer, or zero if β is not integer. The bracket signs around β imply the integer part of β. Coefficients H_{ilm} are defined by:

$$H_{ilm} = \frac{\mathrm{d}}{\mathrm{d}t^m} \left\{ t^i \left[\sum_{k=2}^{J-[\beta]-l+1} \frac{t^k}{k!} g^{(k)}(0) \right]^l \right\} (t=0). \tag{A 4}$$

Note that the finite part concept is fundamental to the present formulation, see Appendix B for additional arguments on this point.

In the present case, the kernel K is defined by:

$$K(t, g(t), \epsilon) = \frac{1}{t^2} \frac{\epsilon + g(t)}{([\epsilon+g(t)]^2 + t^2)^{\frac{1}{2}}}, \tag{A 5}$$

where variable t and function $g(t)$ are linked to the spanwise location and the lifting-line geometry by:

$$t = y - \psi, \tag{A 6}$$

$$g(t) = x_0(y) - x_0(\psi). \tag{A 7}$$

As a result, formula (A 3) must be applied up to order $J=0$ with $\beta = -2$ and $S(t) = \operatorname{sgn}(t)$. Subsequent calculations present no difficulties.

Appendix B

The finite part integral concept was introduced by Hadamard (1932) when he was working on his extensive theory of the Cauchy problem. Some of the relevant aspects of Hadamard's finite part concept are summerized here.

Let a be a positive real and $F(x)$ be a function locally integrable on the interval $[0, a]$. Furthermore, $F(x)$ is assumed to be singular at the origin such that it may be decomposed into the sum of a function $G(x)$, locally integrable throughout the entire interval $[0, a]$, and a generalized polynomial $P(x)$ such that:

$$P(x) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} P_{ij} x^{p_i} \ln^j (x). \tag{B 1}$$

The exponents p_i are less than or equal to minus one. Under these hypotheses, it is possible to prove the following identity for all small, positive real ϵ:

$$\int_\epsilon^a F(x)\,\mathrm{d}x = \mathrm{FP}(\epsilon) - I^\infty(\epsilon), \tag{B 2}$$

$I^\infty(\epsilon)$ is the indefinite integral of $P(x)$ evaluated at ϵ; it is also a generalized polynomial with respect to ϵ. $I^\infty(\epsilon)$ diverges to infinity when ϵ tends to zero; for this reason it is called the infinite part of the integral. $\mathrm{FP}(\epsilon)$ is composed of all the terms which remain finite when ϵ tends to zero. The limit of $\mathrm{FP}(\epsilon)$, when ϵ tends to zero, is called the finite part of the integral; it is denoted $\mathrm{FP}\int_0^a F(x)\,\mathrm{d}x$. Put in mathematical terms, this definition yields:

$$\mathrm{FP}\int_0^a F(x)\,\mathrm{d}x = \lim_{\epsilon\to 0} \mathrm{FP}(\epsilon). \tag{B 3}$$

A similar definition can be set for functions which are not integrable in the vicinity of infinity. This case is treated by the change of variables: $x = 1/t$.

From these definitions some interesting properties can be derived. Let $\phi(x)$ be a function which is smooth in the vicinity of the origin and n be a strictly positive integer, then it may be easily verified that

$$\mathrm{FP}\int_0^a \frac{\phi(x)}{x^n}\,\mathrm{d}x = \lim_{\epsilon\to 0}\left\{\int_\epsilon^a \frac{\phi(x)}{x^n}\,\mathrm{d}x + \phi^{(0)}(0)\frac{\epsilon^{1-n}}{1-n} + \phi^{(1)}(0)\frac{\epsilon^{2-n}}{2-n} + \ldots + \phi^{(n-1)}(0)\frac{\ln(\epsilon)}{(n-1)!}\right\}. \tag{B 4}$$

If instead of an integer, n, the exponent of x is a real number, λ, greater than one, then an equivalent result can be obtained. The term $\ln(\epsilon)$, however, has to be replaced by $\epsilon^{k-1+\lambda}/(k+1-\lambda)$, where k is the integer part of $\lambda - 1$. Result (B 4) hints that the finite part concept is a suitable tool for evaluating asymptotic expansions of integrals (see Guermond (1987, 1988)). In order to illustrate this idea, some examples are given.

Indeed, (A 3), which constitutes the main result of Appendix A, can be obtained in a very classical way using the MAE technique. For this purpose, a small positive quantity, δ, is defined such that

$$\delta = o(1), \tag{B 5}$$

$$\epsilon = o(\delta). \tag{B 6}$$

Then, the domain of integration is subdivided into two domains such that:

$$I(\epsilon) = \int_{D-[-\delta,+\delta]} f(t)K(t, g(t), \epsilon)\,\mathrm{d}t + \int_{-\delta}^{+\delta} f(t)K(t, g(t), \epsilon)\,\mathrm{d}t. \tag{B 7}$$

The first integral is called the outer contribution and the second one is called the inner contribution. In the first integral, ϵ is much smaller than variable t throughout the domain of integration, hence, kernel K can be uniformly approximated by its

Taylor expansion with respect to ϵ. Then it is a simple matter of algebra to show that the outer contribution can be put into the form

$$\int_{D-[-\delta,+\delta]} f(t)\,K(t,g(t),\epsilon)\,\mathrm{d}t = \sum_{j=0}^{J} \mathrm{FP}\int_D f(t)\,\partial_3^j K(t,g(t),0)\,\mathrm{d}t\frac{\epsilon^j}{j!}+P(\delta)+o(\epsilon^J), \quad \text{(B 8)}$$

where $P(\delta)$ is a diverging generalized polynomial of the form, (B 1).

In the integral representing the inner contribution, variable t is of order ϵ throughout the domain of integration. Variable t is conveniently changed into $\epsilon\tilde{t}$ and functions $f(\epsilon\tilde{t})$, $g(\epsilon\tilde{t})$, $K(\epsilon\tilde{t}, g(\epsilon\tilde{t}), \epsilon\tilde{t})$ are replaced by their Taylor expansion with respect to ϵ. After some calculation, the inner contribution yields the second part of the right-hand side of (A 3) minus the diverging polynomial $P(\delta)$ together with terms which are $o(\epsilon^J)$. Hence the final result is obtained.

In order to test the present technique, the reader can apply it on the following simple example. Let a be a negative real and b be a positive real. Let $\phi(x)$ be a smooth function on interval $[a,b]$, then it is easy to verify the following asymptotic expansion for small values of ϵ:

$$\int_a^b \frac{\phi(x)}{x^2+\epsilon^2}\mathrm{d}x = \frac{\phi^{(0)}(0)}{|\epsilon|}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{x^2+1}+\mathrm{FP}\int_a^b\frac{\phi(x)}{x^2}\mathrm{d}x$$
$$+|\epsilon|\tfrac{1}{2}\phi^{(2)}(0)\,\mathrm{FP}\int_{-\infty}^{+\infty}\frac{x^2}{x^2+1}\mathrm{d}x+o(\epsilon). \quad \text{(B 9)}$$

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